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COMPUTATIONAL ASPECTS OF THE PROBLEM OF OPTIMAL FLIGHT AS A BOUNDARY PROBLEM

by

V. K. Isayev and V. V. Sonin

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ABSTRACT: There is proposed [1] a modification of the Newton method for solving boundary problems for ordinary differential equation. In this article there is examined one example of applying this method to the problem of optimal motion of a point of variable mass M(t) in a central gravitational field with a limited expenditure of power. English Translation: 14 pages, 10 equations, 4 figures and 2 tables.

COMPUTATIONAL ASPECTS OF THE PROBLEM OF OPTIMAL FLIGHT AS A BOUNDARY PROBLEM

V. K. Isayev and V. V. Sonin

(Moscow)

1. Work [1] proposed a modification of the Newton method for solving boundary problems for ordinary differential equations. In this article we examine one example of applying this method to the problem of optimal motion of a point of variable mass M(t) in a central gravitational field with a limited expenditure of power.

For convenience let us briefly reproduce the derivation of the basic relationships (for more detail, see [2]). Let the allowable velocity $\underline{c}$ of the discharged mass be limited $0 < c_{min} \leq c_{max} < \infty$, and let the power of the reactive jet $N = -\dot{M}c^2/2$ lie within the limits $0 \leq N \leq N_{max}$ for all values of $\underline{c}$. The flat motion of a mass point in vacuum in a Newtonian gravitational field is described by the system

$$\dot{u} = \frac{Nu_1 \cos\varphi}{mc} - \frac{\mu x}{(x^2+y^2)^{3/2}},$$
$$\dot{v} = \frac{Nu_1 \sin\varphi}{mc} - \frac{\mu y}{(x^2+y^2)^{3/2}} \quad (1)$$
$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{m} = -\frac{Nu_1}{c^2},$$

where $u_1 = N/N_{max}$, $N = 2N_{max}/M(0)$, $m(t) = M(t)/M(0)$; x, y – the Cartesian coordinates of the point of variable mass, u, v – components of its

velocity, φ — the angle between the Ox axis and the direction of the thrust vector. We shall also search for the programs of change in power $u_1(t)$ $(0 \leq u_1(t) \leq 1)$, the velocity of the reactive jet $c(t)$ and the direction of the thrust vector $\varphi(t)$, which during the time $t = T$ move the material point with an initial mass $M(0)$ from the position $x(0) = x^0$, $y(0) = u^0$, $u(0) = v^0$ to the position $x(T) = x^1$, $y(T) = y^1$, $u(T) = u^1$, $v(T) = v^1$ under conditions of minimal expenditure of mass or equivalent conditions of maximal final mass.

2. To solve the stated problem with the aid of the maximum principle of L. S. Pontryagin, let us compose the function

$$H = p_u\left[\frac{Nu_1\cos\varphi}{mc} - \frac{\mu x}{(x^2+y^2)^{3/2}}\right] + p_v\left[\frac{Nu_1\sin\varphi}{mc} - \frac{\mu y}{(x^2+y^2)^{3/2}}\right] + p_x u + p_y v - p_m\frac{Nu_1}{c^2},$$

where p_i $(i = u, v, x, y, m)$ are the conjugate variables that satisfy the system of differential equations

$$\dot{p}_i = -\frac{\partial H}{\partial i} \qquad (i = u, v, x, y, m). \tag{2}$$

The condition of the maximum of the function H relative to the control functions (u_1, c, φ) for the given bounds gives values of the max--optimal equations that are introduced in Table 1.

Table 1

Number of the regime	Condition of realization of the regime	Values of the optimal equations			
		u_1	c	$\cos\varphi$	$\sin\varphi$
0	$\frac{\sqrt{p_u^2+p_v^2}}{m} + \frac{p_m}{c_{max}} < 0$	0	—		
1	$\frac{\sqrt{p_u^2+p_v^2}}{m} + \frac{2p_m}{c_{min}} > 0$	1	c_{min}		
2	$\frac{\sqrt{p_u^2+p_v^2}}{m} < -\frac{2p_m}{c_{min}} < \frac{c_{max}}{c_{min}} \times \frac{\sqrt{p_u^2+p_v^2}}{m}$	1	$-\frac{2mp_m}{\sqrt{p_u^2+p_v^2}}$	$-\frac{p_u}{\sqrt{p_u^2+p_v^2}}$	$-\frac{p_v}{\sqrt{p_u^2+p_v^2}}$
3	$-p_m < c_{max}\frac{\sqrt{p_u^2+p_v^2}}{m} < -2p_m$	1	c_{max}		

Putting the values of the optimal equations found from the table into the initial (1) and conjugate (2) systems, we reduce the variational problem formulated above to a boundary problem of the 10-th order with these conditions:

$$u(0) = u^0, \quad v(0) = v^0, \quad x(0) = x^0, \quad y(0) = y^0; \quad m^0 = 1; \tag{3}$$

$$u(T) = u^1, \quad v(T) = v^1, \quad x(T) = x^1, \quad y(T) = y^1,$$
$$p_m(T) = p_m^1 = -1. \tag{4}$$

The variational problem was reduced to calculating $p_u(0)$, $p_v(0)$, $p_x(0)$, $p_y(0)$, $p_m(0)$ from conditions (4), which were fulfilled on the right end.

We shall consider that the values u(T), v(T), x(T), y(T), $p_m(T)$ are well approximated by the continuous and differentiable functions of the sought initial conditions. Let there be an approximation $p_i(0) = p_i^0$ (i = u, v, x, y, m) that corresponds to the solution of the system of equations (1), (2) that satisfies conditions (3), but does not satisfy conditions (4). If there are small perturbations of the sought approximation

$$\bar{p}_i(0) = p_i^0 + \delta p_i^0 \quad (i = u, v, x, y, m), \tag{5}$$

because of which the Cauchy problem for the system of equations (1), (2) with the initial conditions (3), (5) satisfies conditions (4), i.e.,

$$\begin{aligned}
u(T, p_u^0 + \delta p_u^0, p_v^0 + \delta p_v^0, p_x^0 + \delta p_x^0, p_y^0 + \delta p_y^0, p_m^0 + \delta p_m^0) &= u^1,\\
v(T, p_u^0 + \delta p_u^0, p_v^0 + \delta p_v^0, p_x^0 + \delta p_x^0, p_y^0 + \delta p_y^0, p_m^0 + \delta p_m^0) &= v^1,\\
x(T, p_u^0 + \delta p_u^0, p_v^0 + \delta p_v^0, p_x^0 + \delta p_x^0, p_y^0 + \delta p_y^0, p_m^0 + \delta p_m^0) &= x^1,\\
y(T, p_u^0 + \delta p_u^0, p_v^0 + \delta p_v^0, p_x^0 + \delta p_x^0, p_y^0 + \delta p_y^0, p_m^0 + \delta p_m^0) &= y^1,\\
p_m(T, p_u^0 + \delta p_u^0, p_v^0 + \delta p_v^0, p_x^0 + \delta p_x^0, p_y^0 + \delta p_y^0, p_m^0 + \delta p_m^0) &= p_m^1,
\end{aligned} \tag{6}$$

then, by disregarding the terms higher than the first order, we can write (6) in the form

$$\frac{\partial u}{\partial p_u^0}\delta p_u^0+\frac{\partial u}{\partial p_v^0}\delta p_v^0+\frac{\partial u}{\partial p_x^0}\delta p_x^0+\frac{\partial u}{\partial p_y^0}\delta p_y^0+\frac{\partial u}{\partial p_m^0}\delta p_m^0=$$
$$=u^1-u(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0),$$
$$\frac{\partial v}{\partial p_u^0}\delta p_u^0+\frac{\partial v}{\partial p_v^0}\delta p_v^0+\frac{\partial v}{\partial p_x^0}\delta p_x^0+\frac{\partial v}{\partial p_y^0}\delta p_y^0+\frac{\partial v}{\partial p_m^0}\delta p_m^0=$$
$$=v^1-v(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0),$$
$$\frac{\partial x}{\partial p_u^0}\delta p_u^0+\frac{\partial x}{\partial p_v^0}\delta p_v^0+\frac{\partial x}{\partial p_x^0}\delta p_x^0+\frac{\partial x}{\partial p_y^0}\delta p_y^0+\frac{\partial x}{\partial p_m^0}\delta p_m^0=$$
$$=x^1-x(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0), \qquad (7)$$
$$\frac{\partial y}{\partial p_u^0}\delta p_u^0+\frac{\partial y}{\partial p_v^0}\delta p_v^0+\frac{\partial y}{\partial p_x^0}\delta p_x^0+\frac{\partial y}{\partial p_y^0}\delta p_y^0+\frac{\partial y}{\partial p_m^0}\delta p_m^0=$$
$$=y^1-y(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0),$$
$$\frac{\partial p_m}{\partial p_u^0}\delta p_u^0+\frac{\partial p_m}{\partial p_v^0}\delta p_v^0+\frac{\partial p_m}{\partial p_x^0}\delta p_x^0+\frac{\partial p_m}{\partial p_y^0}\delta p_y^0+\frac{\partial p_m}{\partial p_m^0}\delta p_m^0=$$
$$=p_m^1-p_m(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0).$$

In order to use system (7), we must calculate the values $u(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)$, $v(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)$, $x(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)$, $y(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)$, $p_m(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)$, that stand in the right parts of (7) and we must calculate the matrix of the derivatives.

We shall obtain the first by solving the Cauchy problem for the system (1), (2) with the initial conditions (3) and

$$p_u(0)=p_u^0, \qquad p_v(0)=p_v^0, \qquad p_x(0)=p_x^0, \qquad p_y(0)=p_y^0,$$
$$p_m(0)=p_m^0. \qquad (8)$$

To calculate each column of the matrix of derivatives, we must solve the system of equations in variations or (additionally) we must solve the Cauchy problem for system (1), (2).

For example, the first column of the matrix is determined from the formulas

$$\frac{\partial u}{\partial p_u^0}=\frac{u(T,p_u^0+\Delta p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)-u(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)}{\Delta p_u^0},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial p_m}{\partial p_u^0}=\frac{p_m(T,p_u^0+\Delta p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)-p_m(T,p_u^0,p_v^0,p_x^0,p_y^0,p_m^0)}{\Delta p_u^0}. \qquad (9)$$

In order to use formulas (9), we must solve the Cauchy problem for the system (1), (2) with the initial given conditions (3) and

$$p_u(0) = p_u^0 + \Delta p_u^0, p_v(0) = p_v^0, \quad p_x(0) = p_x^0, \quad p_y(0) = p_y^0, \quad p_m(0) = p_m^0.$$

The remaining 4 columns of the matrix can be calculated in a similar manner.

To solve the Cauchy problem, we used the Runge-Kutta method with automatic selection of the step with respect to accuracy. During calculation of the derivatives we took

$$\Delta p_i^0 = \begin{cases} 10^{-3} p_i^0, & |p_i^0| \geq 1; \\ 10^{-3}, & |p_i^0| < 1 \end{cases} \quad (i = u, v\ x, y, m).$$

After determining the numerical values of the parameters of system (7), let us find the increments δp_i^0 (i = u, v, x, y, m) from it. Taking $p_i^0 + \delta p_i^0$ (i = u, v, x, y, m) as the initial point, let us repeat the procedure of determining the parameters of the "new" system (7) and calculating the "new" δp_i^0.

However, the clasical Newtonian method introduced above, as it applies to the examined problem, usually does not converge. To improve the convergence, according to the terminology of work [1], let us introduce the "miss function," i.e., the magnitude that characterizes the quality of fulfillment of conditions (4). (In the given example the distance between the actual end of the phase trajectory and the point (4) was taken as this magnitude. The use of this function allows us to correct the found increments δp_i^0 (i = u, v, x, y, m), which improves the convergence.)

For this purpose, after determining δp_i^0 (i = u, v, x, y, m) let us calculate the functions

$$\begin{aligned} &u(T, p_u + \alpha\delta p_u^0,\ p_v^0 + \alpha\delta p_v^0,\ p_x^0 + \alpha\delta p_x^0,\ p_y^0 + \alpha\delta p_y^0,\ p_m^0 + \alpha\delta p_m^0) = \\ &\qquad = u(T, p^0 + \alpha\delta p^0), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &p_m(T, p_u^0 + \alpha\delta p_u^0,\ p_v^0 + \alpha\delta p_v^0,\ p_x^0 + \alpha\delta p_x^0,\ p_y^0 + \alpha\delta p_y^0,\ p_m^0 + \alpha\delta p_m^0) = \\ &\qquad = p_m(T, p^0 + \alpha\delta p^0), \end{aligned} \tag{10}$$

solving for the series of fixed values α the Cauchy problem for systems

(1), (2) with the initial conditions (3) and $p_u(0) = p_u^0 + \alpha\delta p_u^0$, $p_v(0) = p_v^0 + \alpha\delta p_v^0$, $p_x(0) = p_x^0 + \alpha\delta p_x^0$, $p_y(0) = p_y^0 + \alpha\delta p_y^0$, $p_m(0) = p_m^0 + \alpha\delta p_m^0$. From the values (10) let us find the "miss function" $\Phi(\alpha)$

$$\Phi^2(\alpha) = [u(T, p^0 + \alpha\delta p^0) - u^1]^2 + [v(T, p^0 + \alpha\delta p^0) - v^1]^2 + \\ + [x(T, p^0 + \alpha\delta p^0) - x^1]^2 + [y(T, p^0 + \alpha\delta p^0) - y^1]^2 + \\ + [p_m(T, p^0 + \alpha\delta p^0) - p_m^1]^2$$

and the value α^* that corresponds to the minimum $\Phi(\alpha)$ in the interval $0 < \alpha \leq 1$. From the preliminarily calculated values of $\Phi(\frac{1}{2})$ and $\Phi(1)$, let us find the parabola that approximates $\Phi(\alpha)$:

$$\Phi(\alpha) \approx L_1(\alpha) = \Phi(0) - \left[3\Phi(0) - 4\Phi\left(\frac{1}{2}\right) + \Phi(1)\right]\alpha + \\ + 2\left[\Phi(0) - 2\Phi\left(\frac{1}{2}\right) + \Phi(1)\right]\alpha^2.$$

The parabola $L_1(\alpha)$ has one extremum at the point

$$\alpha_* = \frac{3\Phi(0) - 4\Phi\left(\frac{1}{2}\right) + \Phi(1)}{4\left[\Phi(0) - 2\Phi\left(\frac{1}{2}\right) + \Phi(1)\right]}.$$

Let us examine the following separately:

1) the case of the minimum $\Phi(0) - 2\Phi(\frac{1}{2}) + \Phi(1) > 0$: a) $\alpha_* \geq 1$, b) $0 < \alpha_* < 1$, c) $\alpha_* \leq 0$;

2) the case of the maximum $\Phi(0) - 2\Phi(\frac{1}{2}) + \Phi(1) < 0$: a) $\alpha_* \geq 1$, b) $0 < \alpha_* < 1$, c) $\alpha_* \leq 0$.

In cases 1,a) and 2,c) (and with the additional condition that $\Phi(1) < \Phi(0)$ is fulfilled, also in case 2,b)) we should set $\alpha^* = 1$; in case 1,b), we should take $\alpha^* = \alpha_*$; in remaining cases we should repeat the process in the lesser interval $0 \leq \alpha \leq \frac{1}{2}$, i.e., we should calculate $\Phi(\frac{1}{4})$, from the magnitudes of $\Phi(0)$, $\Phi(\frac{1}{4})$ and $\Phi(\frac{1}{2})$, we should construct the parabola $L_2(\alpha)$ that approximates $\Phi(\alpha)$ in the interval $0 \leq \alpha \leq \frac{1}{2}$, etc. If in this case we do not manage to find the value α^*, we should calculate $\Phi(\frac{1}{3})$, construct the polynominal $L_3(\alpha)$ for the

interval $0 \leqq \alpha \leqq \frac{1}{8}$, etc., until the process leads to a certain value of α^*. After α^* is found, the iterative process is renewed from point (3) and

$$p_u(0) = p_u^0 + \alpha^* \delta p_u^0, \quad p_v(0) = p_v^0 + \alpha^* \delta p_v^0,$$
$$p_x(0) = p_x^0 + \alpha^* \delta p_x^0, \quad p_y(0) = p_y^0 + \alpha^* \delta p_y^0,$$
$$p_m(0) = p_m^0 + \alpha^* \delta p_m^0.$$

3. Let us make a number of comments connected with the realization of the described algorithm on a computer.

With this method a Cauchy problem is solved for a system of 10-th order equations (1), (2) for different initial conditions at each step of the iteration not less than eight times (six times to obtain the right parts and the matrix of the derivatives of system (7) and twice to determine $\Phi(\frac{1}{2})$ and $\Phi(1)$), a system of linear 5-th order equations is solved, a series of logical operations is carried out, etc.

As computational practice shows, at least 90% of the machine time is spent on solving the Cauchy problem. Therefore, to reduce the machine time it is expedient at the start of the iterative search, when the "miss" with respect to the boundary conditions is still comparatively great, to solve the Cauchy problem with a relatively large constant step or with a relatively large error in the step (during calculation with automatic selection of the step), decreasing the step as $\Phi(1)$ is reduced (or, correspondingly, the error that is allowable in the step).

While the accuracy of the calculation will not increase in the future, the computational error will become comparable with the magnitude of the miss and the process will cease to converge. Hence it is evident that it is necessary to control the accuracy of the solution to a system of differential equations in the interval [0, T].

We used the following method*: system (1) and (2) has two first integrals (see [2]) $H = \text{const}$ and $M = p_u v - p_v u + p_x y - p_y x = \text{const}$. On the ends of the integration interval we calculated the values $M(0)$, $H(T)$, $M(T)$, after which the larger of the magnitudes

$$\frac{|H(0)-H(T)|}{|H(0)|}, \quad \frac{|M(0)-M(T)|}{|M(0)|}$$

took into account the relative error ε of the solution in the interval $[0, T]$. If the relative error became less than ε, then the accuracy of calculation on the following iteration step increased.

4. Let us examine a specific example of the solution of the boundary problem for the following values of the parameters and the boundary conditions**:

$$\begin{array}{lll}
N = 17.506613, & u^0 = 0.00, & u^1 = 0.00 \\
c_{\min} = 0.6, & v^0 = 6.28318531, & v^1 = 5.09015941, \\
c_{\max} = 1000, & x^0 = 1.00, & x^1 = 1.52369100, \\
\mu = 39.4784176, & y^0 = 0.00, & y^1 = 0.00, \\
T = 0.9 & m^0 = 1.00, & p_m^1 = -1.00.
\end{array}$$

As the initial approximation of p^0 we selected the point $p_u^0 = 0.119420502$, $p_v^0 = 0.182035475$; $p_x^0 = 1.16399559$, $p_y^0 = 0.63037049$, $p_m^0 = 0.504237974$.

The course of iteration is easily followed from Table 2 and Figs. 1 -- 3 (given are the values of the phase coordinates, the variables that are connected to them, the miss function Φ and the first integrals H and M when $t = 0$ and $t = T$ depending upon the number of iterations $\underline{k}$).

Figure 4 introduces certain coefficients of the matrix of the system of equations (5) as a function of the parameter $\underline{k}$ (for conveniene

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*The use of the first integrals to evaluate the accuracy of the solution of a system of ordinary differential equations on the interval was used previously by V. A. Yegorov (see, for example, [3]).

**The problem of flight between a circular orbit of earth and Mars that is optimum with respect to expenditure of the working mass (the heliocentric angle of flight $\psi = 360°$, duration $T = 0.9$ year) reduces to the given boundary problem.

discrete values on Figs. 1 – 4 are connected by a smooth curve). The intervals between whole neighboring values of the parameter k on Fig. 3 conditionally correspond to the full lengths of the step in the classical Newtonian method, and the internal points describe the behavior of the miss function in the process of "step subdivision." The dotted line represents the interpolation Lagrangian polynominal $L_1(\alpha)$ (in this variant it is not necessary to calculate $L_2(\alpha)$). From Figs. 1 – 4 it is evident that there is a divergence of the classical Newtonian method in the second-third steps of the iteration (the change in the character of the behavior of the matrix elements shown on Fig. 4 is evidently connected with this), and when $k > 7$ the values of all the magnitudes stabilize. The materials in Table 2 reflect the role of the control of calculation accuracy in the process of solving the boundary problem.

In the 11-th, 12-th, 13-th and 14-th steps of iteration, when the error in calculating the first integrals becomes comparable to $f(k, \alpha^*)$, the solution of system (1), (2) is repeated for the same initial data, but with a greater accuracy in the step. A comparison of the first integrals shows that the required degree of accuracy $\varepsilon = 10^{-7}$ is retained on the resultant trajectory.

Submitted
24 February 1964

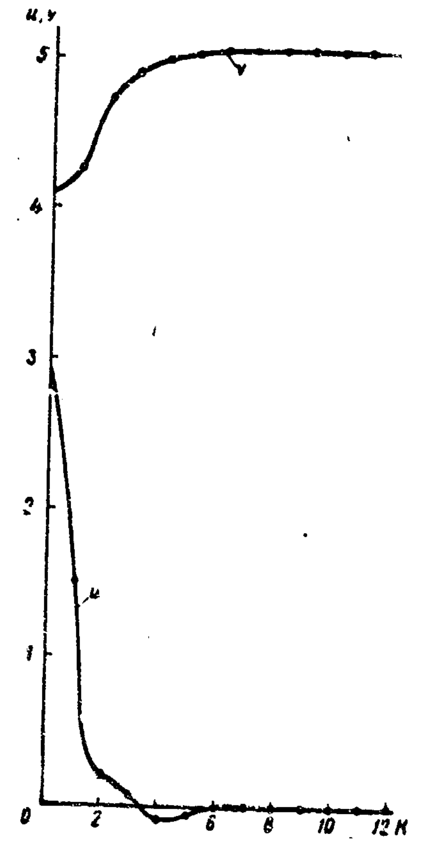

Fig. 1

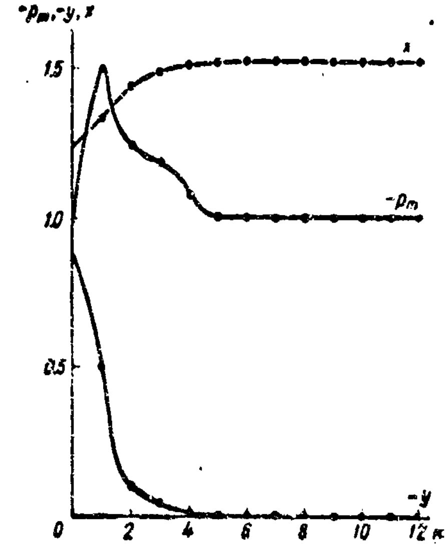

Fig. 2

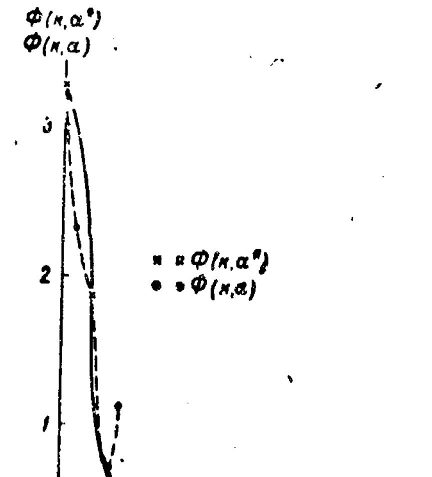

Fig. 3

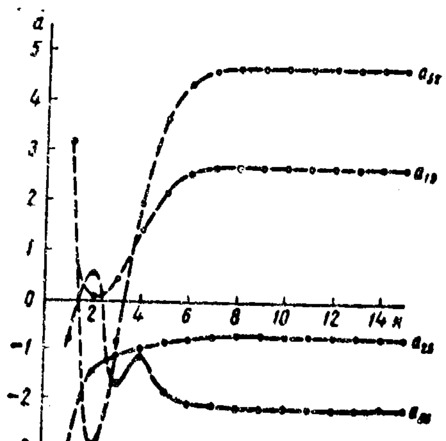

Fig. 4

Table 2

		$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$t=0$	H	0.119970544	0.120151244	0.109314308	0.102198888	0.0872721535	0.0818657218	0.0820675318
	M	-1.57660514	-1.72552062	-1.78931418	-1.76211546	-1.58365007	-1.47390363	-1.45964432
	p_u	0.119420502	0.136904422	0.111892351	0.107233820	0.0938019022	0.0867244685	0.0857830667
	p_v	0.182035475	0.316793994	0.288243379	0.252625773	0.217727647	0.198964416	0.195421333
	p_m	-0.501237974	-0.107190730	-0.764470773	-0.588364019	-0.475200855	-0.429743919	-0.422349721
	p_x	1.16399559	1.68844710	1.56699373	1.41325928	1.22658647	1.12806190	1.11223606
	p_y	0.630370199	0.734389743	0.612575624	0.571661077	0.502121429	0.463040185	0.456923373
$t=T$	H	0.119970224	0.120149860	0.109313548	0.102108399	0.0872717174	0.0818653428	0.0820671551
	M	-1.157660287	-1.72551203	-1.78931187	-1.76211288	-1.58364853	-1.47390228	-1.45964204
	u	2.99189549	1.54845713	0.202794797	0.0746248376	-0.0819788354	-0.0560547297	-0.00951483911
	v	4.11894140	4.26384651	4.72892020	4.9[illegible]844149	5.00468153	5.05815268	5.08399515
	m	0.710098379	0.871926066	0.787432406	0.7[illegible]356093	0.666255609	0.653507431	0.649547208
	x	1.23268906	1.32959672	1.43624773	1.48806486	1.50535598	1.51599508	1.52224125
	y	-0.895598760	-0.484452348	-0.089214444	-0.0417252166	-0.00892371733	-0.000480726913	-0.000221151214
	p_u	0.0167792118	0.125980687	0.171484304	0.169886849	0.158325856	0.144112599	0.141070251
	p_v	-0.259714997	-0.183874653	-0.208487987	-0.223975818	-0.202710858	-0.192003633	-0.192473806
	p_m	-0.999996518	-1.40993116	-1.23291828	-1.17589856	-1.07052149	-1.00625621	-1.001037093
	p_x	-0.532937077	-0.850800467	-1.15659391	-1.12415816	-1.00297518	-0.927817121	-0.914693044
	p_y	0.976292782	0.857777902	0.589792141	0.535655498	0.458650577	0.420033633	0.416165469
	$\Phi(0)$	3.28378399	1.8764122	0.491402047	0.262690754	0.139341613	0.0653082714	0.0114785009
	$\Phi\left(\frac{1}{2}\right)$	2.28090142	0.774460994	0.251762159	0.157973858	0.100703897	0.0386118640	0.00692230185
	$\Phi(1)$	1.87641220	1.12338558	0.548582770	0.166998317	0.0653082696	0.0114786099	0.00235041196
	x^*	1.00000000	0.629753804	0.473352780	0.710329072	1.0000000	1.0	1.0

Table 2

		$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=11$ b and c
$t=0$	H	0.0820762363	0.0820819785	0.0822831831	0.0820834156	0.0825034598	0.0820834598
	M	−1.45700619	−1.45652650	−1.45643673	−1.43641983	−1.43641665	−1.45641665
	p_u	0.0856019432	0.0855684543	0.0855621606	0.055609747	0.0853607512	0.0855607512
	p_v	0.194758351	0.19463,512	0.194606302	0.194601733	0.194000871	0.194600871
	p_m	−0.420863775	−0.420578017	−0.420523896	−0.420513304	−0.420311665	−0.420511665
	p_x	1.10923615	1.10866779	1.10856048	1.10854023	1.10853642	1.10853642
	p_y	0.455776636	0.455560476	0.455519727	0.455512044	0.452510595	0.455510995
$t=T$	H	0.0820758604	0.0820816026	0.0829828072	0.0820830306	0.0820830838	0.0820834116
	M	−1.45700481	−1.45652512	−1.45643535	−1.45641845	−1.45641597	−1.45641628
	u	−0.001[illegible]97447	−0.00037506[illegible]52	$-0.709412200\cdot10^{-4}$	$-0.134008552\cdot10^{-6}$	$-0.251523621\cdot10^{-}$	$-0.862915613\cdot10^{-8}$
	v	5.088[illegible]122	5.08292091	5.09011431	5.099015091	5.09015781	5.09015985
	m	0.648686397	0.648510046	0.648476178	0.648469765	0.648468555	0.648468051
	x	1.52339434	1.52363423	1.52368096	1.52368897	1.52369061	1.52269225
	y	−0.0000412488122	$-0.782318378\cdot10^{-5}$	$-0.147794956\cdot10^{-5}$	$0.275981559\cdot10^{-6}$	$0.544664242\cdot10^{-}$	$-0.340680359\cdot10^{-6}$
	p_u	0.140538518	0.14[illegible]434308	0.147414546	0.140410816	0.140410111[illegible]	0.140409885
	p_v	−0.192531621	−0.1[illegible]552069	−0.192556254	−0.192557052	−0.192557204	−0.192557635
	p_m	−1.00016544	−1.00003001	−1.00000560	−1.00000105	−1.00000019	−1.00000199
	p_x	−0.912275522	−0.911814884	−0.911727857	−0.911711444	−0.911708345	−0.911707697
	p_y	0.413369843	0.415226189	0.415199298	0.415194234	0 415193282	0.415193621
	$\Phi(0)$	0.00235041085	0.000449159987	$0.849506152\cdot10^{-4}$	$0.169397007\cdot10^{-4}$		$0.894136662\cdot10^{-5}$
	$\Phi\left(\frac{1}{2}\right)$	0.00140010389	0.000267070834	$0.504985650\cdot10^{-4}$	$0.952907708\cdot10^{-5}$		$0.446062138\cdot10^{-5}$
	$\Phi(1)$	0.000449156171	$0.849518856\cdot10^{-4}$	$0.169383925\cdot10^{-4}$	$0.301964105\cdot10^{-5}$		$0.215080824\cdot10^{-6}$
	α^*	1.0	1.0	1 0	1.0		1.0

Table 2

		$k=12$	$k=12$ b and e	$k=13$	$k=13$	$k=14$	$k=15$ b and e
$t=0$	H	0.0820831677	0.0820831677	0.0820831319	0.0820831319	0.0820831230	0.0820831231
	M	−1.45641403	−1.45641403	−1.45641371	−1.45641371	−1.45641364	−1.45641364
	p_u	0.0855606262	0.0855606262	0.0855606100	0.0855606100	0.0855606077	0.0855606077
	p_v	0.194600652	0.194600652	0.194600642	0.194600642	0.194600641	0.194600641
	p_m	−0.420511024	−0.420511024	−0.420511005	−0.420511005	−0.420511004	−0.420511004
	p_x	1.10853488	1.10853488	1.10853472	1.10853472	1.10853470	1.10853470
	p_y	0 45510102	0.455510102	0.455510036	0.455510036	0.455510030	0.455510030
$t=T$	H	0.0820831196	0.0820831584	0.0820831224	0.0820831311	0.0820831222	0.0820831230
	M	−1.45641367	−1.45641388	−1.45641365	−1.45641370	−1.45641364	−1.45641364
	u	$-0.111185727\cdot 10^{-8}$	$-0.182952417\cdot 10^{-9}$	$-0.6268188064\cdot 10^{-8}$	$-0.522411937\cdot 10^{-9}$	$0.348946827\cdot 10^{-8}$	$0.382947291\cdot 10^{-7}$
	v	5.09015942	5.09016038	5.09015942	5.09015940	5.09015941	5.09015941
	m	0.648468203	0.640468089	0.648468198	0.648468182	0.648468188	0.648468198
	x	1.52369100	1.52369101	1.52369099	1.52369108	1.52369100	1.52369100
	y	$-0.416491730\cdot 10^{-9}$	$0.264645850\cdot 10^{-9}$	$0.758188179\cdot 10^{-9}$	$0.533385673\cdot 10^{-7}$	$0.282285195\cdot 10^{-9}$	$-0.352085294\cdot 10^{-8}$
	p_u	0.140409898	0.140409931	0.140409900	0.140109889	0 140409899	0.140109898
	p_v	−0 192557126	−0.192557213	−0.192557119	−0.192557134	−0 192557118	−0.192557119
	p_m	−1.0	−1.00000038	−0.999999999	−1.0	−1.0	−1.0
	p_x	−0.911707120	−0.911707391	−0.911707095	−0.911707052	−0.911707091	−0.911707088
	p_y	−0.415192876	0.415192865	0.415192879	0.415192911	0.415192876	0.415192878
	$\Phi(0)$		$0.108336693\cdot 10^{-8}$		$0.534227549\cdot 10^{-6}$		
	$\Phi(\frac{1}{2})$		$0.541146535\cdot 10^{-9}$		$0.259324049\cdot 10^{-9}$		
	$\Phi(1)$		$0.238197530\cdot 10^{-9}$		$0.109566411\cdot 10^{-7}$		
	a^*		1.0		1.0		

REFERENCES

1. V. K. Isayev and V. V. Sonin. Ob odnoy modifikatsii metoda N'yutona chislennogo resheniya krayevykh zadach. Zh. vychisl. matem. i matem. fiz., 3, No. 6, 114-1116, 1963.

2. V. K. Isayev and V. V. Sonin. Ob odnoy nelineynoy zadache optimal'nogo upravleniya. Avtomatika i telemekhan., No. 9, 23, 1117-1129, 1962.

3. V. A. Yegorov. O nekotorykh zadachakh dinamiki poleta k Lune. Uspekhi fiz. nauk, 63, vyp. 1a, 73-117, 1957.